and $\mu^*_{k-1}(1) = \text{continue}$. **Q.E.D.**

Thus the optimum policy is to continue until the δ^{th} object, where δ is the minimum integer such that $\left(\frac{1}{N-1} + \cdots + \frac{1}{\delta}\right) \leq 1$, and then stop at the first time an element is observed with largest rank.

Exercise 4.19

a) Let the state $x_k \in \{T, \bar{T}\}$ where T represents the driver having parked before reaching the kth spot. Let the control at each parking spot $u_k \in \{P, \bar{P}\}$ where P represents the choice to park in the kth spot. Let the disturbance w_k equal 1 if the kth spot is free; otherwise it equals 0. Clearly, we have the control constraint that $u_k = \bar{P}$, if $x_k = T$ or $w_k = 0$. The cost associated with parking in the kth spot is:

$$g_k(\bar{T}, P, 1) = k$$

If the driver has not parked upon reaching his destination, he incurs a cost $g_N(\bar{T}) = C$. All other costs are zero. The system evolves according to:

$$x_{k+1} = \begin{cases} T, & \text{if } x_k = T \text{ or } u_k = P \\ \bar{T}, & \text{otherwise} \end{cases}$$

Once the driver has parked, his remaining cost is zero. Thus, we can define F_k to be the expected remaining cost, given that the driver has not parked before the kth spot. (Note that this is simply $J_k(\bar{T})$). The DP algorithm is given by:

$$\begin{aligned} F_0 &= C \\ F_k &= \min_{u_k \in \{P, \bar{P}\}} \mathop{E}_{w_k} \{ g_k(\bar{T}, u_k, w_k) + J_{k-1}(x_{k-1}) \} \\ F_k &= \min \Big[\underbrace{p\big[k + J_{k-1}(T)\big]}_{park,\ free} + \underbrace{qJ_{k-1}(\bar{T})}_{park,\ not\ free} \ , \ \underbrace{J_{k-1}(\bar{T})}_{don't\ park} \Big] \end{aligned}$$

But since $J_i(T) = 0 \quad \forall i$:

$$\begin{aligned} F_k &= \min\big[pk + qF_{k-1}, F_{k-1}\big] \qquad (1) \\ &= p\min\big[k, F_{k-1}\big] + qF_{k-1} \end{aligned}$$

b) From (1), we see that the driver should stop when $F_{k-1} > k$. Assume it is optimal to stop at the $(\text{k+1})^{\text{st}}$ spot. Then $F_k > k+1 > k$. Now, we also see from (1) that $F_k \leq F_{k-1} \quad \forall k$. Thus, $F_{k-1} > k$ and it is also optimal to stop at the kth spot. Now assume that it is optimal not to stop at the kth spot. Then

$$F_k \leq F_{k-1} \leq k \leq k+1$$

and it is optimal not to stop at the (k+1)st spot.

Thus, there exists some k^* where it is optimal to continue if $k \geq k^*$ and it is optimal to park if $k < k^*$. In particular, k^* is the smallest integer satisfying $k^* \geq F_{k^*-1}$. Since $F_{k^*} \leq F_0 = C$ we know that such a $k^* < \infty$ exists.

Claim:

$$F_k = k + q^k C - \frac{q}{p}(1-q^k) \qquad \text{whenever} \quad k < F_{k-1}$$

Proof: The proof follows by induction. For $k=1$,

$$F_1 = \; p\min[1, C] + qC \; = \; p + qC$$

where the claim gives $\quad F_1 = 1 - \frac{q}{p}(1-q) + qC = p + qC$

Assume the claim for $k = i$. Then:

$$\begin{aligned}
F_{i+1} &= p\min\big[i+1, F_i\big] + qF_i \\
&= p(i+1) + q\left[i + q^i C - \frac{q}{p}(1-q^i)\right] \quad \text{using} \quad i+1 < F_i \\
&= i + 1 - q + q^{i+1}C - \frac{q}{p}\big[q(1-q^i)\big] \\
&= i + 1 + q^{i+1}C - \frac{q}{p}\big[q(1-q^i) + p\big] \\
&= i + 1 + q^{i+1}C - \frac{q}{p}(1-q^{i+1})
\end{aligned}$$

Since $k^* - 1 < F_{k^*-2}$ we can determine k^* as the smallest integer satisfying:

$$k^* \geq F_{k^*-1} = k^* - 1 + q^{k^*-1}C - \frac{q}{p}(1-q^{k^*-1})$$

Rearranging this inequality yields:

$$\frac{1}{p} \geq q^{k^*-1}C + \frac{q^{k^*}}{p}$$

or finally,

$$(q+pC)^{-1} \geq q^{k^*-1}$$